Convergence in Measure and the LP Spaces

The problems below are taken out of various textbooks on real variables, including "Real Analysis" by Elias M. Stein and Rami Shakarchi and "Real Analysis" by N. L. Carothers. Questions are also taken from real variables qualifying exams at CUNY Graduate Center. The problems are color-coded. The color green indicates that the problem came from a textbook and to the best of my knowledge was not featured on any qualifying exam. Yellow means that the problem was spotted in at least one qualifying exam. Red indicates that the problem or one just like it appeared in at least two qualifying exams.

1. Show that $m\{|f - g| \geq \epsilon\} \leq m\{|f - h| \geq \epsilon/2\} + m\{|h - g| \geq \epsilon/2\}$. Thus, the expression $m\{|f - g| \geq \epsilon\}$ behaves rather like a metric.

2. Prove that limits in measure are unique up to equality a.e. That is, if $\{f_n\}$ converges in measure to both f and g, then f = g a.e.

3. If $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$, prove that $f_n + g_n \xrightarrow{m} f + g$.

4. If $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$, does it follow that $f_n g_n \xrightarrow{m} fg$? If not, what additional hypotheses are needed?

5. True or false? If $f_n \xrightarrow{m} f$, then $|f_n| \xrightarrow{m} |f|$.

6. Prove or give a counterexample. If the statement is false, what corrections are needed to make it true?
   (a) If $f_n \to f$ almost uniformly (a.u.), then $f_n \xrightarrow{m} f$.
   (b) If $f_n \to f$ a.u. then $f_n \to f$ a.e.
   (c) If $f_n \to f$ pointwise a.e., then $f_n \xrightarrow{m} f$.
   (d) If $f_n \xrightarrow{m} f$, then $f_n \to f$ pointwise a.e.
   (e) If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{m} f$.
   (f) If $f_n \xrightarrow{m} f$, then $f_n \xrightarrow{L^1} f$.

7. Prove the **Riesz-Fisher Theorem** for Cauchy sequences in measure. Namely, show that if $\{f_n\}$ is Cauchy in measure, then there is some function f such that $f_n \xrightarrow{m} f$. Moreover, $\{f_n\}$ contains a subsequence $\{f_{n(k)}\}$, which converges to f pointwise a.e.

8. Prove that Fatou's lemma holds for convergence in measure: If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n \xrightarrow{m} f$, show that $f \geq 0$ a.e. and that $\int f \leq \liminf_{n\to\infty} \int f_n$

9. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbf{R}^d$ with $|f_n| \leq g$, for all n, where $g \in L^1(\mathbf{R}^d)$. If $\{f_n\}$ converges to f in measure, prove that $|f| \leq g$ a.e. and that $\{f_n\}$ converges to f in L^1. In other words, prove that the dominated convergence theorem holds for convergence in measure.

10. Let $\{f_n\}$, $\{g_n\}$, and g be integrable on $\mathbf{R}^d$, and suppose that $f_n \xrightarrow{m} f$, $g_n \xrightarrow{m} g$, $|f_n| \leq g_n$ a.e., for all n, and that $\int g_n \to \int g$. Prove that $f \in L^1$ and that $\int f_n \to \int f$. (Compare with exercise 20 in the Lebesgue Integration problem list).

11. Let $1 < p < \infty$ and define q by the equation $1/p + 1/q = 1$. Prove

(a) **Young's Inequality.** Suppose that $1/p + 1/q = 1$. Then, for any $a, b \geq 0$, we have $ab \leq a^p/p + b^q/q$, with equality occurring if and only if $a^{p-1} = b$.

(b) **Hoelder's Inequality.** Given that $1/p + 1/q = 1$, $f \in L^p(E)$, and $g \in L^q(E)$

$$\left|\int_E fg\right| \leq \int_E |fg| \leq \|f\|_p \|g\|_q$$

(c) **Minkowski's Inequality.** Let f, $g \in L^p(E)$. Then $f + g \in L^p(E)$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. Consequently, $\|\cdot\|_p$ is a norm.

12. Suppose that $m(E) < \infty$.

(a) If $1 \leq p < q < \infty$, show that $L^q(E) \subset L^p(E)$.

(b) Under the assumptions in part (a), show that $\|f\|_p \leq (m(E))^{1/p-1/q}\|f\|_q$. In particular, if E = [0, 1], notice that the L^p-norms increase with p; that is, $\|f\|_p \leq \|f\|_q$ for $1 \leq p < q < \infty$.

13. Given $1 \leq p < q < \infty$, show that $L^p(\mathbf{R}) \neq L^q(\mathbf{R})$ by showing that neither containment holds. That is, construct functions $f \in L^q(\mathbf{R}) - L^p(\mathbf{R})$ and $g \in L^p(\mathbf{R}) - L^q(\mathbf{R})$.

14. Given $1 \leq p, q, r < \infty$ with $r^{-1} = p^{-1} + q^{-1}$, prove the following generalization of Hoelder's inequality: $\|fg\|_r \leq \|f\|_p \|g\|_q$ whenever $f \in L^p$ and $g \in L^q$.

15. Supply a proof for the following:

(a) **Liapounov's inequality.** Given $1 \le p, q < \infty$ and $0 \le \alpha \le 1$, let $r = \alpha p + (1-\alpha)q$. Then $\|f\|_r^r \le \|f\|_p^{\alpha p} \|f\|_q^{(1-\alpha)q}$.

(b) Suppose that $1 \le p < r < q < \infty$. Then $L^p \cap L^q \subset L^r$.

(c) For $1 \le p < r < q < \infty$, $L^r \subset L^p + L^q$. That is, each $f \in L^r$ is the sum of a function in L^p and a function in L^q.

16. Let $f \in L^2([0, 1])$ and $\int_0^1 f^2 \le 1$.

(a) Show that for each $t \in (0, 1]$, we have $\int_0^t |f| \le \sqrt{t}$.

(b) Show that $\lim_{t\to 0} t^{-1/2} \int_0^t |f| = 0$.

17. If $\{f_n\}$ converges to f in L^p, does $\{|f_n|^p\}$ converge to $|f|^p$ in L^1? in measure?

18. Given $1 \le p < \infty$, construct $f, g \in L^p(\mathbf{R})$ such that $fg \notin L^p(\mathbf{R})$. Thus, although L^p is a vector space and a lattice under the usual pointwise a.e. operations on functions, it is not typically an algebra of functions.

19. Prove the **Riesz-Fisher Theorem** for Cauchy sequences in L^p. Namely, show that if $\{f_n\}$ is Cauchy in L^p, then there is some function f such that $f_n \xrightarrow{L^p} f$. Moreover, $\{f_n\}$ contains a subsequence $\{f_{n(k)}\}$, which converges to f pointwise a.e.

20. Suppose that $\{f_n\}$ is in L^p, $1 \le p < \infty$, with $\|f_n\|_p \le 1$ and $f_n \to f$ a.e. Prove that $f \in L^p$ and that $\|f\|_p \le 1$.

21. For $1 < p < \infty$ and $a, b \ge 0$, show that $a^p + b^p \le (a+b)^p \le 2^{p-1}(a^p + b^p)$ and that the reverse inequalities hold when $0 < p < 1$.

22. Let f, $f_n \in L^p$, $1 \le p < \infty$, and suppose $f_n \to f$ pointwise a.e. Show that $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$. Note that the result also holds if "a.e." is replaced by "in measure."

23. It makes perfect sense to consider the spaces L^p for $0 < p < 1$. In this range, the expression $\|\cdot\|_p$ no longer defines a norm; nevertheless, L^p is a complete metric linear space. For $0 < p < 1$, prove that:

(a) L^p is a vector space.

(b) The expression $d(f,\ g) = \int |f - g|^p$ defines a complete, translation-invariant metric on L^p.

(c) Let $p^{-1} + q^{-1} = 1$ (note that $q < 0$). If $0 \le f \in L^p$ and if $g \ge 0$ satisfies $0 < \int g^q < \infty$, then $\int fg \ge \left(\int f^p\right)^{1/p}\left(\int g^q\right)^{1/q}$.

(d) If f, $g \in L^p$, with f, $g \ge 0$, then $\|f+g\|_p \ge \|f\|_p + \|g\|_p$.

(e) If f, $g \in L^p$, then $\|f+g\|_p \le 2^{1/p}\left(\|f\|_p + \|g\|_p\right)$.

24. Let $f: E \to [-\infty, \infty]$ be measurable and essentially bounded, and let $A = \text{ess.sup}_{x\in E} |f(x)|$. Prove that:

(a) $0 \le A < \infty$ and $|f| \le A$ a.e.

(b) $f = 0$ a.e. if and only if $A = 0$.

(c) If $0 \le A' < A$, then $m\{|f| > A'\} \ne 0$.

Thus, $|f| \le \|f\|_\infty$ a.e., where $\|f\|_\infty$ is the L^∞-norm of f and $\|f\|_\infty$ is the smallest constant with this property.

25. If $f \in L^\infty$, is $m\{|f| = \|f\|_\infty\} > 0$? Is $\{|f| = \|f\|_\infty\} \ne \emptyset$? Explain.

26. If $f: E \to \mathbf{R}$ is measurable, (everywhere) bounded function, prove that $\text{ess.sup}_E |f| \le \sup_E |f|$. Give an example showing that strict inequality can occur.

27. If $f: E \to [-\infty, \infty]$ is essentially bounded, show that

$$ess.\sup_{x\in E} |f(x)| = \inf\left\{\sup_{x\in E-N} |f(x)|:\ \ m(N) = 0\right\}.$$

Moreover, show that this infimum is actually attained; that is, prove that there is a null set N such that $\text{ess.sup}_E |f| = \sup_{E-N} |f|$.

28. Let $f \in C[0, 1]$ and $0 \le A < \infty$. If $|f(x)| \le A$ a.e. $x \in [0, 1]$, prove that, in fact, $|f(x)| \le A$ for all $x \in [0, 1]$. Conclude that

$$\sup_{0\le x\le 1}|f(x)| = ess.\sup_{0\le x\le 1}|f(x)|$$

in this case. In other words, $\|f\|_{C[0,\ 1]} = \|f\|_{L^\infty[0,\ 1]}$.

29. If f, $g: E \to [-\infty, \infty]$ are essentially bounded, show that $f + g$ is essentially bounded and that $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$, where $\|\cdot\|_\infty$ denotes the L^∞-norm.

30. If f, g $\in L^\infty$, show that $fg \in L^\infty$ and $\|f \cdot g\|_\infty \le \|f\|_\infty \|g\|_\infty$. Conclude that L^∞ is a normed algebra. Is L^∞ a normed lattice (under the usual pointwise a.e. ordering)?

31. If E $\subset \mathbf{R}^d$ and m(E) < ∞, show that, as sets, $L^\infty(E) \subset L^p(\mathbf{R}^d)$, for any 1 ≤ p < ∞, and that $\|f\|_p \le m(E)^{1/p}\|f\|_\infty$ for any $f \in L^\infty(E)$. In particular, if $f \in L^\infty[0, 1]$, then $\|f\|_1 \le \|f\|_p \le \|f\|_\infty$ for any 1 ≤ p < ∞.

32. If $f \in L^\infty(E)$, where m(E) < ∞, show that $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$.

33. Suppose $f \in L^\infty(\mathbf{R})$, where the measure on **R** is the usual Lebesgue measure. Prove that

$$\lim_{n\to\infty}\left(\int_R \frac{|f(x)|^n}{1+x^2}dx\right)^{1/n}$$

exists and equals $\|f\|_\infty$.

Solutions:

1. This is immediate when we observe that $\{|f - g| \ge \epsilon\} \subset \{|f - h| \ge \epsilon/2\} \cup \{|h - g| \ge \epsilon/2\}$. The set relationship holds, since any x not in $\{|f - h| \ge \epsilon/2\} \cup \{|h - g| \ge \epsilon/2\}$ satisfies $|f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence, that x cannot be a member of $\{|f - g| \ge \epsilon\}$.

2. It is enough to show that $m\{|f - g| \ne 0\} = 0$, from which it will follow that $|f - g| = 0$ a.e. and hence that f = g a.e. To accomplish this, fix $\epsilon > 0$ and define $E_k = \{|f - g| \ge 1/k\}$. Then for each n, exercise 1 implies that $m(E_k) \le m\{|f_n - f| \ge 1/2k\} + m\{|f_n - g| \ge 1/2k\}$ and, since $f_n \xrightarrow{m} f$ and $f_n \xrightarrow{m} g$, we may choose n large enough so that $m\{|f_n - f| \ge 1/2k\} < \epsilon\, 2^{-k-1}$ and $m\{|f_n - g| \ge 1/2k\} < \epsilon\, 2^{-k-1}$. This shows that $m\{|f - g| \ne 0\} = m(\bigcup_{k=1}^\infty E_k) \le \sum_{k=1}^\infty m(E_k) \le \epsilon \sum_{k=1}^\infty 2^{-k} = \epsilon$. Since ϵ is arbitrary, the desired conclusion is established.

3. By definition, $f_n + g_n \xrightarrow{m} f + g$ if for every $\epsilon > 0$, $\lim_{n\to\infty} m\{|f_n + g_n - f - g| \ge \epsilon\} = 0$. But by exercise 1,

$$m\{|f_n + g_n - f - g| \ge \epsilon\} \le m\{|f_n - f| \ge \epsilon/2\} + m\{|g_n - g| \ge \epsilon/2\}.$$

And since $f_n \xrightarrow{m} f$ and $g_n \xrightarrow{m} g$ by hypothesis, we may use the squeeze theorem to conclude

$$\lim_{n\to\infty} m\{|f_n + g_n - f - g| \ge \epsilon\} \le \lim_{n\to\infty} (m\{|f_n - f| \ge \epsilon/2\} + m\{|g_n - g| \ge \epsilon/2\}) = 0.$$

4. Convergence in measure is not generally preserved by products. Here is a counterexample:

Let $f_n(x) = \dfrac{1+1/n}{1+x^2}$ and $g_n(x) = x^3 + 1/n$. Then $f_n \to f$ and $g_n \to g$, where $f(x) = \dfrac{1}{1+x^2}$, $g(x) = x^3$, and where the mode of convergence is uniform. Note that uniform convergence is stronger and therefore entails convergence in measure. However, $f_n g_n$ does not converge in measure to f g as can be seen by noting that for each $\epsilon > 0$ and every n, m$\{| f_n g_n - fg | \geq \epsilon\}$ = m$\left\{ x \in \mathbf{R}: \left| \dfrac{1}{n}\left(\dfrac{x^3}{1+x^2} + \dfrac{1}{n}\dfrac{1}{1+x^2} \right) \right| \geq \epsilon \right\} = \infty$.

In the counterexample above, notice that the sequence $\{f_n\}$ is uniformly bounded whereas each function in the sequence $\{g_n\}$ is unbounded on $\mathbf{R}$. One way to enforce the product rule for convergence in measure is to add the hypothesis that that both sequences are uniformly bounded outside arbitrarily small sets. Specifically, assume there is some number A such that $\lim_{n\to\infty}$ m{x: $| f_n(x) | \geq$ A} = $\lim_{n\to\infty}$ m{x : $| g_n(x) | \geq$ A} = 0. Then for any $\epsilon > 0$ we can write

$$\begin{aligned}
\lim_{n\to\infty} m\{| f_n g_n - fg | \geq \epsilon\} &\leq \lim_{n\to\infty} m\{| f_n || g_n - g | \geq \epsilon/2\} + \lim_{n\to\infty} m\{| g || f_n - f | \geq \epsilon/2\} \\
&\leq \lim_{n\to\infty} m\{x: | f_n(x) | \geq A\} + \lim_{n\to\infty} m\{| g_n - g | \geq \epsilon/2A\} + \lim_{n\to\infty} m\{x: | g_n(x) | \geq A\} \\
&\quad + \lim_{n\to\infty} m\{| f_n - f | \geq \epsilon/2A\} = 0.
\end{aligned}$$

We are able to conclude from the squeeze theorem that $\lim_{n\to\infty} m\{| f_n g_n - fg | \geq \epsilon\} = 0$.

5. By definition of convergence in measure, we have $| f_n | \xrightarrow{m} | f |$ if and only if for any $\epsilon > 0$, $\lim_{n\to\infty} m\{\big|| f_n | - | f |\big| \geq \epsilon\} = 0$. Since $\big|| f_n(x) | - | f(x) |\big| \leq | f_n(x) - f(x) |$ for every x, it follows that $\{\big|| f_n | - | f |\big| \geq \epsilon\} \subset \{| f_n - f | \geq \epsilon\}$ and because $f_n \xrightarrow{m} f$,

$$\lim_{n\to\infty} m\{\big|| f_n | - | f |\big| \geq \epsilon\} \leq \lim_{n\to\infty} m\{| f_n - f | \geq \epsilon\} = 0.$$

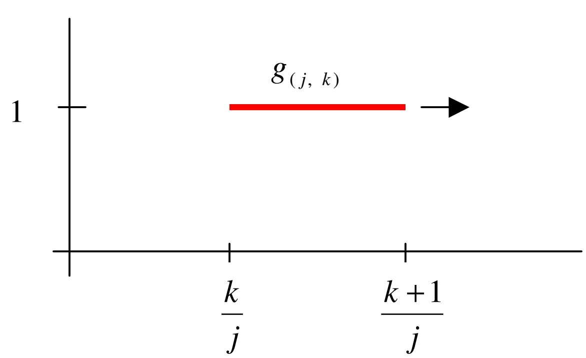
Thus the assertion of the exercise is true.

6. (a) The statement is true. To show this, recall that $f_n \xrightarrow{a.u.} f$ if for every $\epsilon > 0$, there is a measurable set E_ϵ of measure m(E_ϵ) < ϵ such that $f_n \to f$ uniformly for all x $\notin E_\epsilon$. On the other hand, $f_n \xrightarrow{m} f$, if for every $\epsilon > 0$, there is an integer N such that m$\{| f_n - f | \geq \epsilon\} < \epsilon$ for all n ≥ N. Thus, if $f_n \xrightarrow{a.u.} f$, we can find a set $E_{\epsilon/2}$ of measure m($E_{\epsilon/2}$) < $\epsilon/2$ and an integer N so that $| f_n(x) - f(x) | < \epsilon$ for all x $\notin E_{\epsilon/2}$ and all n ≥ N. We then have m$\{| f_n - f | \geq \epsilon\} \leq$ m($E_{\epsilon/2}$) < $\epsilon/2 < \epsilon$, which proves that $f_n \xrightarrow{m} f$.

(b) This statement is also true. Since $f_n \xrightarrow{a.u.} f$, we can find for each k a measurable set E_k of measure $m(E_k) < 2^{-k}$ such that $f_n \to f$ uniformly for all $x \notin E_k$. Define $E = \bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_k$. Then $m(E) \le m(\bigcup_{k=j}^{\infty} E_k) \le \sum_{k=j}^{\infty} 2^{-k}$ and since j is arbitrary, E must be of measure 0. If $x \notin E$, there must be some j for which $x \notin \bigcup_{k=j}^{\infty} E_k$ and in particular there is some E_k, which doesn't contain x. As f_n is uniformly convergent outside of this E_k, it follows that $f_n(x) \to f(x)$. In other words, $f_n \to f$ pointwise on E^c.

(c) Pointwise convergence does not generally imply convergence in measure. Consider, for example, $f_n,\ g_n : \mathbf{R} \to \mathbf{R}$ given by $f_n = \chi_{[n,\ n+1]}$ and $g_n = \chi_{[n,\ \infty)}$. Both functions converge pointwise to 0, but $m\{| f_n | \ge 1\} = m([n, n+1]) = 1$ and $m\{| g_n | \ge 1\}$ $m([n, \infty)) = \infty$. Hence neither f_n nor g_n converge to 0 in measure.
Notice that the function sequences of the counterexample above are defined on a set of infinite measure. If, however, $f_n : E \to \mathbf{R}$ are defined on a set E of finite measure ($m(E) < \infty$) and $f_n \to f$ pointwise a.e., then Egorov's theorem implies that $f_n \xrightarrow{a.u.} f$. By part (a) of this exercise, almost uniform convergence is stronger that convergence in measure.

(d) A sequence of functions can converge in measure and fail to converge pointwise for every x. In order to understand the counterexample below with ease, imagine the graphs of the f_n to be sliding horizontal platforms of vanishing length that move back and forth over [0, 1]. More precisely, define $g_{(j,\ k)} = \chi_{[k/j,\ (k+1)/j]}$, where j, k are nonnegative integers and $0 \le k \le j - 1$. We wish to enumerate this collection of ordered pairs so that the sequence of the graphs of $g_{(j,\ k)}$ plays like the cartoon which is represented by the figure below:



To achieve this, define

$$f_n = g_{(j,\,k)} \text{ if } \sum_{l=1}^{j} l \le n < \sum_{l=1}^{j+1} l \text{ and } k = n - \sum_{l=1}^{j} l.$$

Then $f_n \xrightarrow{m} 0$, because for any $\epsilon > 0$, there is a j such that $1/j < \epsilon$ and therefore m{{| f_n | ≥ ϵ} ≤ $1/j < \epsilon$ for all $n \ge \sum_{l=1}^{j} l$. Notice, however that for any x ∈ [0, 1], $\limsup_{n\to\infty} f_n(x) = 1$ and $\liminf_{n\to\infty} f_n(x) = 0$, and therefore $\lim_{n\to\infty} f_n(x)$ does not exist.

Though $f_n \xrightarrow{m} f$ does not imply pointwise convergence, it is always possible to produce a subsequence $f_{n(k)}$ which converges almost uniformly to f:

First, construct a strictly increasing sequence of integers n(k) such that m{| $f_n - f$ | ≥ 2^{-k}} < 2^{-k} for all n ≥ n(k). Use these integers n(k) as indices of the subsequence $\{f_{n(k)}\}_{k=1}^{\infty}$ and label by E_k the set {| $f_{n(k)} - f$ | ≥ 2^{-k}}. Then m(E_k) < 2^{-k} and for any ϵ > 0, we can therefore select some number j, for which $\sum_{k=j}^{\infty} 2^{-k} < \epsilon$ and define $E_\epsilon = \bigcup_{k=j}^{\infty} E_k$.

Clearly, m(E_ϵ) < ϵ. For all x ∉ E_ϵ, we must have | $f_{n(k)}(x) - f(x)$ | < 2^{-k}, whenever k ≥ j. Thus, if $\delta > 0$, | $f_{n(k)}(x) - f(x)$ | < δ for all k ≥ max {j, $\ln(\delta^{-1})/\ln(2)$}. Hence, convergence is uniform outside E_ϵ.

(e) Convergence in L^1 is stronger than convergence in measure. Recall that $f_n \xrightarrow{L^1} f$ if $\lim_{n\to\infty} \int |f_n - f| = 0$. Given ϵ > 0, Chebyshev's inequality implies that m{| $f_n - f$ | ≥ ϵ} ≤ $\frac{1}{\epsilon}\int |f_n - f|$. Therefore, $\lim_{n\to\infty}$ m{| $f_n - f$ | ≥ ϵ} ≤ $\lim_{n\to\infty} \frac{1}{\epsilon}\int |f_n - f|$ = 0.

(f) This is false. To see this, simply modify the counterexample in part (d); define $f_n = j^2 g_{(j,\,k)}$ if $\sum_{l=1}^{j} l \le n < \sum_{l=1}^{j+1} l$ and $k = n - \sum_{l=1}^{j} l$. Then $f_n \xrightarrow{m} 0$, because for any ϵ > 0, there is a j such that $1/j < \epsilon$ and therefore m{{| f_n | ≥ ϵ} ≤ $1/j < \epsilon$ for all $n \ge \sum_{l=1}^{j} l$.

However, $\int |f_n| \ge j$, for all $n \ge \sum_{l=1}^{j} l$ and therefore $\lim_{n\to\infty} \int |f_n| = \infty$.

The statement can be made true with the additional stipulation that the $\{f_n\}$ are supported on a set E of finite measure and uniformly bounded a.e.:

Let B be an upper bound a.e. of | f_n |, n = 1, 2, … From part (d) we know that convergence in measure implies pointwise convergence via a subsequence. Thus B ≥ | f | as well. Fix ϵ > 0 and define E_n = {| $f_n - f$ | ≥ ϵ}. Select n large enough to insure m(E_n) < ϵ and estimate

$$\int | f_n - f | = \int_{E_n} | f_n - f | + \int_{E-E_n} | f_n - f | \le 2Bm(E_n) + \varepsilon\, m(E) < 2B\varepsilon + \varepsilon\, m(E).$$

Consequently, $f_n \xrightarrow{L^1} f$ as desired.

7. Observe that since $\{f_n\}$ is Cauchy in measure, we can make the set {x $\in \mathbf{R}^d$: $| f_n(x) - f_m(x) | \ge 2^{-k}$} arbitrarily small if we choose large integers m, n. Specifically, let n(k) be a large enough integer so that whenever n, m ≥ n(k), m{$| f_n - f_m | \ge 2^{-k}$} < 2^{-k}. Thus, upon selecting a strictly increasing sequence $\{n(k)\}_{k=1}^{\infty}$, we obtain the subsequence $\{f_{n(k)}\}$ and define E_k = {$| f_{n(k+1)} - f_{n(k)} | \ge 2^{-k}$}. The choice of the n(k) dictates that m(E_k) < 2^{-k} and therefore that $\sum_{k=1}^{\infty} m(E_k) = 1 < \infty$.

Define $f = f_{n(1)} + \sum_{k=1}^{\infty} (f_{n(k+1)} - f_{n(k)})$ and $g = | f_{n(1)} | + \sum_{k=1}^{\infty} | f_{n(k+1)} - f_{n(k)} |$. Then $|f| \le g$, f(x) is defined, and, by construction, is the pointwise limit $f(x) = \lim_{j\to\infty} f_{n(j+1)}(x)$ for all x for which g(x) converges absolutely. Now if it is the case that g(x) = ∞, it must also be true that x $\in E_k$ for infinitely many k. But by the Borel-Cantelli lemma (see problem 10 in the list on measure theory), the set of points in the intersection of infinitely many E_k must have measure 0. That is, if G = {x $\in \mathbf{R}^d$: g(x) = ∞}, then m(G) = 0 and therefore f is the pointwise limit of $\{f_{n(k)}\}$ for all x $\in \mathbf{R}^d - G$.

To see that $f_{n(k)} \xrightarrow{m} f$, pick ϵ > 0 and observe that {$| f - f_{n(k)} | \ge \epsilon$} ⊂ {$| f - f_{n(k)} | \ge 2^{-k+1}$} for all k such that $2^{-k+1} < \epsilon$. Also observe that

$$| f - f_{n(k)} | = \left| \sum_{j=k}^{\infty} (f_{n(j+1)} - f_{n(j)}) \right| \le \sum_{j=k}^{\infty} | f_{n(j+1)} - f_{n(j)} |,$$

which means {$| f - f_{n(k)} | \ge 2^{-k+1}$} ⊂ S_k, where

$$S_k = \left\{ x \in \mathbf{R}^d : \sum_{j=k}^{\infty} | f_{n(j+1)}(x) - f_{n(j)}(x) | \ge 2^{-k+1} \right\}.$$

Furthermore, notice that $S_k \subset \bigcup_{j=k}^{\infty} E_j$, otherwise we would have some x $\in S_k$ for which $\sum_{j=k}^{\infty} | f_{n(j+1)}(x) - f_{n(j)}(x) | < \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1}$, a contradiction. Hence m(S_k) ≤ $\sum_{j=k}^{\infty} m(E_j)$ ≤ 2^{-k+1}. Putting this all together yields

$$\lim_{k\to\infty} m\{| f - f_{n(k)} | \ge \epsilon\} \le \lim_{k\to\infty} m(S_k) \le \lim_{k\to\infty} 2^{-k+1} = 0.$$

Finally, to establish $f_n \xrightarrow{m} f$, for any ϵ > 0, choose k large enough so that

$$m\{| f - f_{n(k)} | \ge \epsilon/2\} < \epsilon/2 \qquad (1)$$

and

$$m\{| f_n - f_{n(k)} | \geq \epsilon/2\} < \epsilon/2 \qquad (2)$$

for all n > n(k).

Inequality (1) can be made valid from the earlier result $f_{n(k)} \xrightarrow{m} f$, whereas inequality (2) comes from the hypothesis that $\{f_n\}$ is Cauchy in measure. For all such n and n(k) we then have (see exercise 1)

$$m\{| f - f_n | \geq \epsilon\} \leq m\{| f - f_{n(k)} | \geq \epsilon/2\} + m\{| f_n - f_{n(k)} | \geq \epsilon/2\} < \epsilon.$$

8. We start by picking a subsequence $\{\int f_{n(k)}\}_{k=1}^{\infty}$ of $\{\int f_n\}_{n=1}^{\infty}$ with the property that $\lim_{k\to\infty} \int f_{n(k)} = \liminf_{n\to\infty} \int f_n$. Since $f_n \xrightarrow{m} f$, it must also be the case that $f_{n(k)} \xrightarrow{m} f$ and by the discussion in exercise 6 part (d), $\{f_{n(k)}\}$ contains a further subsequence $\{f_{n(k[l])}\}$ which converges to f almost uniformly and therefore a.e. From the hypothesis that all the f_n are nonnegative, the pointwise convergence $f_{n(k[l])} \to f$ easily implies $f \geq 0$ a.e. and by Fatou's lemma,

$$\liminf_{l\to\infty} \int f_{n(k[l])} \geq \int f. \qquad (1)$$

However, $\{\int f_{n(k[l])}\}$ is a subsequence of the convergent sequence $\{\int f_{n(k)}\}$ and must therefore go to the same limit. In particular

$$\liminf_{l\to\infty} \int f_{n(k[l])} = \lim_{l\to\infty} \int f_{n(k[l])} = \lim_{k\to\infty} \int f_{n(k)} = \liminf_{n\to\infty} \int f_n. \qquad (2)$$

Combining inequality (1) with the identity chain (2) yields

$$\liminf_{n\to\infty} \int f_n \geq \int f,$$

which is the desired conclusion.

9. Suppose that $f_n \xrightarrow{m} f$ and $| f_n | \leq g$, for all n, where $g \in L^1(\mathbf{R}^d)$. By the argument at the end of exercise 6 part (d), there exists a subsequence $\{f_{n(k)}\}$ of $\{f_n\}$ which converges to f a.u. and hence a.e. In particular, $|f| = \lim_{k\to\infty} | f_{n(k)} | \leq g$. Fix $\epsilon > 0$ and define for each integer N the set $E_N = \{x \in \mathbf{R}^d : |x| \leq N, g(x) \leq N\}$. Then the sequence $g_N = g\chi_{E_N}$ increases monotonically to g a.e. and by the monotone convergence theorem, there must be an integer N large enough so that

$$\int_{E_N{}^c} g = \int (g - g_N) < \varepsilon$$

Now fix k > 0 so that $1/k < \epsilon$ and define $F_n = \{| f_n - f | \geq 1/k\}$. The hypothesis that $f_n \xrightarrow{m} f$ allows us to pick n_0 such that $m(F_n) < \epsilon$ whenever $n \geq n_0$. From our choices of N, k, and n_0 we then have

$$\int | f_n - f | = \int_{E_N} | f_n - f | + \int_{E_N{}^c} | f_n - f | \qquad (1)$$

$$\int_{E_N{}^c} | f_n - f | \leq 2\int_{E_N{}^c} g < 2\varepsilon \tag{2}$$

$$\int_{E_N} | f_n - f | = \int_{E_N \cap F_n} | f_n - f | + \int_{E_N \cap F_n{}^c} | f_n - f | \tag{3}$$

$$\int_{E_N \cap F_n} | f_n - f | \leq 2N\ m(F_n) < 2N\varepsilon \tag{4}$$

$$\int_{E_N \cap F_n{}^c} | f_n - f | \leq (1/k)\ m(E_N) < \varepsilon\ m(E_N) \tag{5}$$

Putting inequalities (1) — (5) together gives the estimate

$$\int | f_n - f | = (2N + m(E_N) + 2)\varepsilon$$

which shows that the integral is arbitrarily small for all $n \geq n_0$. We have thus demonstrated the desired result.

10. Define for each n and N the set $E_N^n = \{x \in \mathbf{R}^d : |x| \leq N,\ g_n(x) \leq N\}$ and set $E_N = \{x \in \mathbf{R}^d : |x| \leq N,\ g_n(x) \leq N$ for all n$\}$. That is $E_N = \bigcap_{n=1}^{\infty} E_N^n$. Several observations are in place.

Observation 1: $g_n \xrightarrow{m} g$, where the g_n are nonnegative. As explained at the end of exercise 6 part (d), there is a subsequence $\{g_{n(k)}\}$ such that $g_{n(k)} \to g$ pointwise a.e. and therefore $0 \leq g(x)$ for almost every x. Furthermore, $g(x) \leq N$ whenever $x \in E_N$.

Observation 2: The sets E_N are increasing ($E_N \subset E_{N+1}$) and since the g_n are integrable, the functions are finite for almost every x. That is, if $U_n = \{x \in \mathbf{R}^d : g_n(x) = \infty\}$ and $U = \{x \in \mathbf{R}^d : g_n(x) = \infty\}$ for at least one n$\}$, then $U = \bigcup_{n=1}^{\infty} U_n$ and $m(U) = 0$. Thus the E_N must increase to a measurable subset $E \subset \mathbf{R}^d$ where $m(\mathbf{R}^d - E) = 0$.

Observation 3: By hypothesis, $|f_n(x)| \leq g_n(x)$ a.e. and $f_n \xrightarrow{m} f$. We may therefore pick subsequences $\{f_{n(k)}\}$ and $\{g_{n(k)}\}$ such that $f_{n(k)} \to f$ a.e. and $g_{n(k)} \to g$ a.e. Therefore $g(x) = \lim_{k\to\infty} g_{n(k)}(x) \geq \lim_{k\to\infty} |f_{n(k)}(x)| = |f(x)|$. In particular, since g is integrable, so must be f and for all $x \in E_N$, $|f_n(x)| \leq N$ and $|f(x)| \leq N$.

Observation 4: For each N

$$\lim_{n\to\infty} \int_{E_N} (g_n - g) = 0$$

This follows from the version of the bounded convergence theorem outlined in exercise 9, in which pointwise convergence a.e. is replaced by convergence in measure. In particular, since $g_n \xrightarrow{m} g$, E_N is a bounded set, and $|g_n(x) - g(x)| \leq 2N$ for all $x \in E_N$, we must have $g_n - g \xrightarrow{L^1} 0$. Furthermore, using observation 3, we may also conclude that

$$\lim_{n\to\infty} \int_{E_N} | f_n - f | = 0$$

Observation 5: By observation 4 and the hypothesis $\int g_n \to \int g$, it follows that

$$\lim_{n\to\infty} \int_{E_N^c} (g_n - g) = \lim_{n\to\infty} \int (g_n - g) - \lim_{n\to\infty} \int_{E_N} (g_n - g) = 0$$

We are now ready to prove the main result by estimating $\left|\int f_n - \int f\right| = \left|\int (f_n - f)\right|$.

$$\left|\int (f_n - f)\right| \le \int | f_n - f | = \int_{E_N} | f_n - f | + \int_{E_N^c} | f_n - f |$$

$$\le \int_{E_N} | f_n - f | + \int_{E_N^c} g_n + \int_{E_N^c} g$$

$$= \int_{E_N} | f_n - f | + \int_{E_N^c} (g_n - g) + 2\int_{E_N^c} g$$

$$\le \int_{E_N} | f_n - f | + | \int_{E_N^c} (g_n - g) | + 2\int_{E_N^c} g$$

Since g is integrable on $\mathbf{R}^d$, the integral of g decays to zero outside a large bounded set. More precisely, for $\varepsilon > 0$ we may pick N large enough so that

$$\int_{E_N^c} g < \frac{\varepsilon}{4}$$

Holding this N fixed, we note from observations 4 and 5 that for large n

$$\int_{E_N} | f_n - f | \le \frac{\varepsilon}{4}$$

and

$$| \int_{E_N^c} g_n - g | \le \frac{\varepsilon}{4}$$

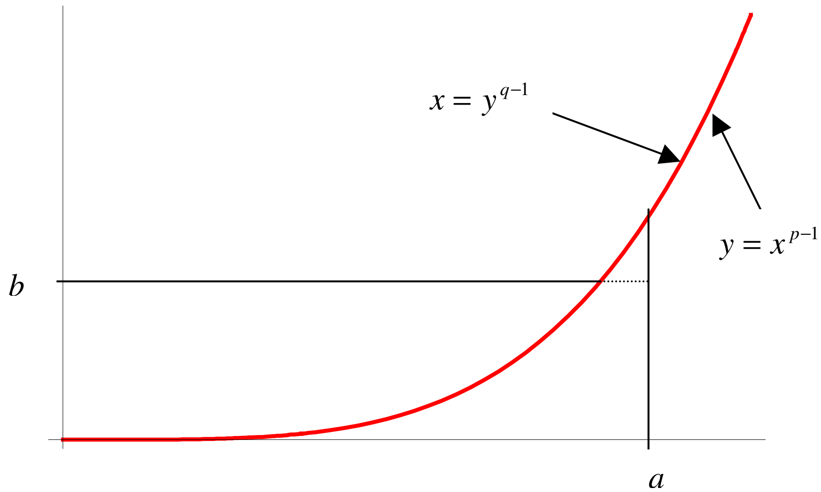
Hence $\left|\int (f_n - f)\right| \le \varepsilon$, from which the assertion $\int f_n \to \int f$ readily follows.

11. The proofs presented below were borrowed from the Carothers textbook.

(a) Notice that if p > 1 and 1/p + 1/q = 1, then q = p/(p−1) > 1. Additionally, notice that q = 1 + 1/(p−1), from which we have q−1 = 1/(p−1). Thus, if p−1 < 1, taking reciprocals establishes that q−1 > 1. Define functions *f* and g : [0, ∞) → **R** by

$$f(x) = x^{p-1} \text{ and } g(y) = y^{1/(p-1)} = y^{q-1}$$

and deduce from the preliminary discussion that f and g are inverses and therefore the graphs y = f(x) and x = g(y) are identical. Without loss of generality p—1 > 1 and the curve y = f(x) is concave up. The diagram below shows that we may think of ab as the area of a rectangle with side-lengths a and b and of $a^p / p + b^q / q$ as the sum of areas between y = f(x) and the x-axis and between x = g(y) and the y-axis.



In other words, $ab \le \int_0^a x^{p-1}dx + \int_0^b y^{q-1}dy = \frac{a^p}{p} + \frac{b^q}{q}$, where equality holds if and only if the corner (a, b) lies on the curve y = f(x). That is, if and only if $b = a^{p-1}$.

(b) That $\left|\int_E fg\right| \le \int_E |fg|$ is clear from the basic properties of Lebesgue integration. To establish $\int_E |fg| \le \|f\|_p \|g\|_q$, notice that if either $\|f\|_p$ or $\|g\|_q$ is 0, the function $|fg| = 0$ a.e. and there is nothing to prove. So assume $\|f\|_p$ $\ne 0$ and $\ne 0$ and consider $\int_E \frac{|fg|}{\|f\|_p \|g\|_q}$. Letting $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$ and applying Young's inequality, we obtain, by monotonicity of Lebesgue integration,

$$\int_E \frac{|fg|}{\|f\|_p \|g\|_q} \le \frac{1}{p}\int_E \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q}\int_E \frac{|g|^q}{\|g\|_q^q} = \frac{1}{p}\frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q}\frac{\|g\|_q^q}{\|g\|_q^q} = 1$$

Hoelder's Inequality is obtained upon multiplying the left-hand-side and the right-hand –side of the inequality by $\|f\|_p \|g\|_q$.

(c) First notice that if f, g $\in L^p$ (E), then

$$\int_E |f+g|^p \le \int_E (2\max\{|f|,\ |g|\})^p \le 2^p\left(\int_E |f|^p + \int_E |g|^p\right).$$

Thus f + g $\in L^p$ (E).

To prove Minkowski's inequality, observe that $|f+g|^{p-1} \in L^q$ (E), where q is defined by the equation 1/p + 1/q = 1. In fact, since q = p/(p−1),

$$\left\| |f+g|^{p-1}\right\|_q = \left(\int_E |f+g|^p\right)^{(p-1)/p} = \|f+g\|_p^{p-1}.$$

Now,

$$\|f+g\|_p^p = \int_E |f+g|^p = \int_E |f+g|\cdot|f+g|^{p-1} \le \int_E |f|\cdot|f+g|^{p-1} + \int_E |g|\cdot|f+g|^{p-1}.$$

Applying Hoelder's inequality to $\int_E |f|\cdot|f+g|^{p-1}$ and $\int_E |g|\cdot|f+g|^{p-1}$ we get

$$\|f+g\|_p^p \le \|f\|_p\|f+g\|_p^{p-1} + \|g\|_p\|f+g\|_p^{p-1} = \left(\|f\|_p + \|g\|_p\right)\cdot\|f+g\|_p^{p-1}.$$

Dividing the last inequality by $\|f+g\|_p^{p-1}$ gives the desired statement.

12. (a) Let $f \in L^q$ (E) and define A = {x ∈ E: $|f| < 1$} and B = {x ∈ E: $|f| \ge 1$}. Then A and B are disjoint and E = A ∪ B. Since 1 ≤ p < q, $|f|^p < 1$ on A and $|f|^p \le |f|^q$ on B. We thus have

$$\int_E |f|^p = \int_A |f|^p + \int_B |f|^p \le \int_A 1 + \int_B |f|^q \le \int_E 1 + \int_E |f|^q = m(E) + \|f\|_q^q < \infty$$

and hence $f \in L^p$ (E).

(b) Assume 1 ≤ p < q < ∞ and let a = q/p and b = q/(q−p). Then 1/a + 1/b = 1 and therefore, by Hoelder's inequality,

$$\|f\|_p^p = \int_E |f|^p = \int_E (1)\cdot|f|^p \le \left(\int_E (1)^{q/(q-p)}\right)^{1-p/q}\left(\int_E \left(|f|^p\right)^{q/p}\right)^{p/q} = (m(E))^{1-p/q}\|f\|_q^p.$$

Upon taking the right and left-hand-side of the inequality to the power 1/p, we obtain the desired statement.

13. Define

$$f(x) = \begin{cases} x^{-1/p} & if \quad x \ge 1 \\ 0 & if \quad x < 1 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x^{-1/q} & if \quad x \in (0,\ 1] \\ 0 & if \quad x \notin (0,\ 1] \end{cases}$$

Then $\int |f(x)|^p = \int_1^\infty x^{-1}dx = \infty$ while 1 ≤ p < q < ∞ implies that

$\int |f(x)|^q = \int_1^\infty x^{-q/p}dx = \dfrac{p}{q-p} < \infty$. Hence $f \in L^q$ (**R**) – L^p (**R**).

On the other hand, $\int |g(x)|^q = \int_0^1 x^{-1}dx = \infty$ while $\int |g(x)|^p = \int_0^1 x^{-p/q}dx = \dfrac{q}{q-p} < \infty$. Hence g $\in L^p$ (**R**) – L^q (**R**).

14. The equation $r^{-1} = p^{-1} + q^{-1}$ is equivalent to $1 = \frac{1}{(p/r)} + \frac{1}{(q/r)}$, which satisfies Hoelder's condition. Thus, $\int |fg|^r \le \left(\int |f|^{r\cdot(p/r)}\right)^{r/p}\left(\int |g|^{r\cdot(q/r)}\right)^{r/q} = \|f\|_p^r\|g\|_q^r$. Raising both sides of the inequality to the power 1/r procures the desired result.

15. (a) Given $r = \alpha p + (1-\alpha)q$, we may write

$$\|f\|_r^r = \int |f|^r = \int |f|^{\alpha p}|f|^{(1-\alpha)q}.$$

The numbers $a = 1/\alpha$ and $b = 1/(1-\alpha)$ are Hoelder's conjugates ($1/a + 1/b = \alpha + (1-\alpha) = 1$) and we may therefore apply Hoelder's inequality to obtain

$$\int |f|^{\alpha p}|f|^{(1-\alpha)q} \le \left(\int |f|^p\right)^{\alpha}\left(\int |f|^q\right)^{1-\alpha} = \left(\int |f|^p\right)^{\alpha p/p}\left(\int |f|^q\right)^{(1-\alpha)q/q} = \|f\|_p^{\alpha p}\|f\|_q^{(1-\alpha)q}.$$

This validates Liapunov's inequality.

(b) Let $1 \le p < r < q < \infty$ and $f \in L^p \cap L^q$. Then for some $\alpha \in (0, 1)$, $r = \alpha p + (1-\alpha)q$. By Liapunov's inequality, we then have

$$\|f\|_r^r \le \|f\|_p^{\alpha p}\|f\|_q^{(1-\alpha)q} < \infty.$$

Thus $f \in L^r$ and the claim $L^p \cap L^q \subset L^r$ is therefore confirmed to be true.

(c) We will show that for every measurable subset $E \subset \mathbf{R}^d$ and $1 \le p < r < q < \infty$, the relationship $L^r(E) \subset L^p(E) + L^q(E)$ holds. So, for $f \in L^r(E)$, define $A = \{x \in E: |f(x)| \ge 1\}$ and $B = \{x \in E: |f(x)| < 1\}$. Then A and B are disjoint measurable subsets of E with $A \cup B = E$. Hence, $f = f\chi_A + f\chi_B = f_1 + f_2$ and it only remains to be shown that $f_1 \in L^p(E)$ and $f_2 \in L^q(E)$. We have

$$\int_E |f_1|^p = \int_A |f|^p \le \int_A |f|^r \le \int_E |f|^r < \infty,$$

where we use the fact that $|f(x)|^p \le |f(x)|^r$ for all $x \in A$. Therefore the assertion $f_1 \in L^p(E)$ is valid. The estimation

$$\int_E |f_2|^q = \int_B |f|^q \le \int_B |f|^r \le \int_E |f|^r < \infty$$

shows that the corresponding assertion for f_2 is true as well.

16. (a) By letting $g = \chi_{[0,\, t]}$, we can write the integral $\int_0^t |f|$ as $\int_0^1 |f \cdot g|$. Applying Hoelder's inequality with $1/p = 1/q = ½$, we obtain

$$\int_0^1 |f\cdot g| \le \left(\int_0^1 f^2\right)^{1/2}\left(\int_0^1 g^2\right)^{1/2} = \|f\|_2\left(\int_0^t 1\right)^{1/2} = \|f\|_2 \cdot \sqrt{t}.$$

Since $\int_0^1 f^2 \le 1$, we must also have $\left(\int_0^1 f^2\right)^{1/2} = \|f\|_2 \le 1$. This completes the proof of (a).

(b) Fix t ∈ (0, 1] and observe that by the Hoelder Inequality,
$\int_0^t |f| \le \left(\int_0^t |f|^2\right)^{1/2} \left(\int_0^t 1\right)^{1/2} = \left(\int_0^t |f|^2\right)^{1/2} t^{1/2}$. Hence

$$0 \le t^{-1/2}\int_0^t |f| \le \left(\int_0^t |f|^2\right)^{1/2}.$$

An easy application of the Lebesgue Monotone Convergence Theorem shows that $\lim_{t\to 0}\int_0^t f^2 = 0$. Therefore

$$0 \le \lim_{t\to 0} t^{-1/2}\int_0^t |f| \le \left(\lim_{t\to 0}\int_0^t |f|^2\right)^{1/2} = 0,$$

which establishes that $\lim_{t\to 0} t^{-1/2}\int_0^t |f| = 0$.

17. Recall that $f_n \xrightarrow{L^p} f$ if $\left(\int |f_n - f|^p\right)^{1/p} = \|f_n - f\|_p \to 0$ as n → ∞. The expression $\|f_n - f\|_p$ may be vanishing without the assumption that the $f_n \in L^p$. Consequently, it would be best to analyze the behavior of $\lim_{n\to\infty}\left\| |f_n|^p - |f|^p \right\|_1 = \lim_{n\to\infty}\int \left| |f_n|^p - |f|^p \right|$ in two separate cases.

Case 1: Suppose $\{f_n\} \subset L^p$ (E). Then, since $\|f_n - f\|_p \to 0$, we can deduce from Minkowski's inequality that $\|f\|_p \le \|f - f_n\|_p + \|f_n\|_p < \infty$. Hence $f \in L^p$ (E) as well. Moreover, since $\|\cdot\|_p$ is a metric, it follows that $\left| \|f_n\|_p - \|f\|_p \right| \le \|f_n - f\|_p \to 0$. In particular, $\left(\int_E |f_n|^p\right)^{1/p} \to \left(\int_E |f|^p\right)^{1/p}$ and therefore $\int_E |f_n|^p \to \int_E |f|^p$ as n → ∞. Notice that the inequality $\left| \|f_n\|_p - \|f\|_p \right| \le \|f_n - f\|_p$ holds on any subset F ⊂ E. To put this observation to use, define A_n = {x ∈ E: $|f_n(x)|^p \ge |f(x)|^p$} and B_n = {x ∈ E: $|f_n(x)|^p <$ $|f(x)|^p$}. Clearly E is the disjoint union of A_n and B_n and we may write

$$\int_E \left| |f_n|^p - |f|^p \right| = \int_{A_n}\left(|f_n|^p - |f|^p \right) + \int_{B_n}\left(|f|^p - |f_n|^p \right).$$

But we know from Minkowski's inequality that

$$\left(\int_{A_n} | f_n |^p\right)^{1/p} - \left(\int_{A_n} | f |^p\right)^{1/p} = \left|\left(\int_{A_n} | f_n |^p\right)^{1/p} - \left(\int_{A_n} | f |^p\right)^{1/p}\right| \le \left\| f_n - f \right\|_{L^p(A_n)}$$

And since

$$\left\| f_n - f \right\|_{L^p(A_n)} \le \left\| f_n - f \right\|_{L^p(E)} \to 0,$$

the limit

$$\lim_{n\to\infty}\left(\left(\int_{A_n} | f_n |^p\right)^{1/p} - \left(\int_{A_n} | f |^p\right)^{1/p}\right) = 0.$$

This limit is of the form $\lim_{n\to\infty}(x_n - y_n) = 0$, where the hypothesis that $f \in L^p(E)$ implies that the y_n and therefore the y_n are bounded and since the function $g(u) = u^p$ is uniformly continuous over any bounded set, we may further conclude that $\lim_{n\to\infty}(x_n^p - y_n^p) = 0$. That is

$$\lim_{n\to\infty}\int_{A_n}\left(| f_n |^p - | f |^p\right) = 0.$$

The same argument shows that

$$\lim_{n\to\infty}\int_{B_n}\left(| f |^p - | f_n |^p\right) = 0.$$

We are thus lead to the conclusion that $f_n \xrightarrow{L^p} f$ implies $| f_n |^p \xrightarrow{L^1} | f |^p$ under the condition of membership in L^p. Since L^1 convergence is stronger than convergence in measure, note that we also have $| f_n |^p \xrightarrow{m} | f |^p$.

Case 2: Suppose that $\{f_n\}$ is not a sequence of elements in L^p. Then $f \notin L^p$, for otherwise we would have

$$\left\| f_n \right\|_p \le \left\| f - f_n \right\|_p + \left\| f \right\|_p < \infty$$

by Minkowski's inequality. The example below is the idea of Joseph Gunther. It shows that we may not generally expect $| f_n |^p \xrightarrow{L^1} | f |^p$ to follow from the mere hypothesis that $\left\| f_n - f \right\|_p \to 0$. In other words, the assumption $\{f_n\} \subset L^p$ is necessary. The essence behind Joseph's idea is to define $f_n = f + g_n$ for some nonnegative sequence of functions g_n so that $\left\| f_n - f \right\|_2 = \left\| g_n \right\|_2 \to 0$, while $\left\| (f_n)^2 - (f)^2 \right\|_1 = \int (g_n^2 + 2 g_n f) \to \infty$. Therefore, consider the space $L^2(\mathbf{R})$ and define

$$f(x) = \begin{cases} x^{-2/3} & x \in (0,\ 1] \\ 0 & otherwise \end{cases} \quad \text{and} \quad f_n(x) = \begin{cases} x^{-2/3} & x \in (1/n,\ 1] \\ x^{-2/3} + x^{-1/3} & x \in (0,\ 1/n] \\ 0 & otherwise \end{cases}$$

Then

$$\int | f_n - f |^2 = \int_0^{1/n} x^{-2/3} dx = \frac{3}{n^{1/3}} \to 0.$$

But

$$\int \left((f_n)^2 - (f)^2 \right) = \int_0^{1/n} \left((x^{-2/3} + x^{-1/3})^2 - x^{-4/3} \right) dx \geq \int_0^{1/n} x^{-1} = \infty$$

and $| f_n |^2$ does not converge to $| f |^2$ as we set out to show.

18. Consider the functions

$$f(x) = \begin{cases} x^{-1/(3p)} & x \in (0,\ 1] \\ 0 & otherwise \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^{-2/(3p)} & x \in (0,\ 1] \\ 0 & otherwise \end{cases}.$$

Then $\int | f |^p = \int_0^1 x^{-1/3} dx = 3/2$ and $\int | g |^p = \int_0^1 x^{-2/3} dx = 3$. In particular, f and g are elements of L^p (**R**). However, $\int | f \cdot g |^p = \int_0^1 x^{-1} dx = \infty$ and therefore the function $fg \notin L^p$ (**R**).

19. Let $\{ f_n \}$ be Cauchy in L^p. Then for every $\epsilon > 0$ there is some N such that $\| f_n - f_m \|_p <$ ϵ whenever m, n $\geq$ N. We may therefore pick a subsequence $\{ f_{n(k)} \}$ of $\{ f_n \}$ with the property that $\| f_{n(k+1)} - f_{n(k)} \|_p < 2^{-k}$ and define

$$f = f_{n(1)} + \sum_{j=1}^{\infty} (f_{n(j+1)} - f_{n(j)}) \qquad \text{and} \qquad g = | f_{n(1)} | + \sum_{j=1}^{\infty} | f_{n(j+1)} - f_{n(j)} |.$$

Also observe that

$$f_{n(k)} = f_{n(1)} + \sum_{j=1}^{k-1} (f_{n(j+1)} - f_{n(j)})$$

Then $|f| \leq$ g and by Minkowski's inequality,

$$\|g\|_p \leq \|f_{n(1)}\|_p + \sum_{j=1}^{\infty} \| f_{n(j+1)} - f_{n(j)} \|_p \leq \|f_{n(1)}\|_p + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Hence g < ∞ a.e. and therefore the series f is absolutely convergent for almost every x. In particular, $f_{n(k)} \to f$ pointwise a.e. To show that $f_{n(k)} \xrightarrow{L^p} f$ as well, simply apply Minkowski's inequality to obtain the estimate

$$\| f - f_{n(k)} \|_p = \left\| \sum_{j=k}^{\infty} (f_{n(j+1)} - f_{n(j)}) \right\|_p \leq \sum_{j=k}^{\infty} \| f_{n(j+1)} - f_{n(j)} \|_p \leq \sum_{j=k}^{\infty} 2^{-j},$$

which shows that $\| f - f_{n(k)} \|_p$ vanishes as k → ∞.

Finally, given $\epsilon > 0$, select N_1 so that $\|f - f_{n(k)}\|_p < \epsilon/2$ for all $k \geq N_1$, N_2 so that $\|f_n - f_m\|_p < \epsilon/2$ for all m, n $\geq N_2$ and define N = max $\{N_1, N_2\}$. Then for all k, n $\geq$ N, we have $\|f - f_n\|_p \leq \|f - f_{n(k)}\|_p + \|f_{n(k)} - f_n\|_p < \epsilon/2 + \epsilon/2 = \epsilon$. This shows that $f_n \xrightarrow{L^p} f$. Therefore, we have demonstrated that L^p is complete.

20. This is a simple application of Fatou's lemma: Since $\|f_n\|_p \leq 1$, it follows that $\int |f_n|^p = \|f_n\|_p^p \leq 1$ and since $f_n \to f$ a.e., we must have $|f_n|^p \to |f|^p$ a.e. and therefore, by Fatou's lemma,

$$\int |f|^p \leq \liminf_{n\to\infty} \int |f_n|^p \leq 1.$$

This estimate shows at once that $f \in L^p$ and that $\|f\|_p \leq 1$.

21. Observe that $a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p + b^p)$ if and only if it is true that $1 \leq \dfrac{(a+b)^p}{a^p + b^p} \leq 2^{p-1}$, where the last inequality may be expressed as $1 \leq \dfrac{(1+b/a)^p}{1+(b/a)^p} \leq 2^{p-1}$ upon dividing the numerator and denominator of the middle term by a^p. Similarly, the reverse inequality holds if and only if $1 \geq \dfrac{(1+b/a)^p}{1+(b/a)^p} \geq 2^{p-1}$ holds. Without loss of generality, $a \geq b$ and we are therefore lead to consider the function $\varphi(x) = \dfrac{(1+x)^p}{1+x^p}$, where $0 \leq x \leq 1$. This function is continuous for all $0 \leq x \leq 1$ and differentiable in the interval (0, 1) with derivative $\varphi'(x) = \dfrac{p(1+x)^{p-1}(1-x^{p-1})}{(1+x^p)^2}$.

Case 1 ($1 < p < \infty$): In this situation, $\varphi'(x) > 0$ and therefore $\varphi(x)$ is increasing, with minimum $\varphi(0) = 1$ and maximum $\varphi(1) = 2^{p-1}$. Thus $1 \leq \varphi(b/a) \leq 2^{p-1}$, which proves that $a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p + b^p)$.

Case 2 ($0 < p < 1$): Here $\varphi'(x) < 0$ and therefore $\varphi(x)$ is decreasing, with minimum $\varphi(1) = 2^{p-1}$ and maximum $\varphi(0) = 1$. Thus $2^{p-1} \leq \varphi(b/a) \leq 1$, which proves that $a^p + b^p \geq (a+b)^p \geq 2^{p-1}(a^p + b^p)$.

22. One direction is easy. Suppose $\|f_n - f\|_p \to 0$. Then, since $p \geq 1$, the p-norm defines a metric and we have $\left|\|f_n\|_p - \|f\|_p\right| \leq \|f_n - f\|_p \to 0$. From this, it follows by the squeeze theorem that $\|f_n\|_p \to \|f\|_p$.

Now assume $\|f_n\|_p \to \|f\|_p$ holds. From exercise 21, we see that $\|f_n - f\|_p^p = \int |f_n - f|^p \leq \int 2^{p-1}\left(|f_n|^p + |f|^p\right)$. Define functions $k_n = |f_n - f|^p$, $k = 0$, $h_n = 2^{p-1}\left(|f_n|^p + |f|^p\right)$, and $h = 2^p |f|^p$. Then the hypothesis $f_n \to f$ a.e. implies that $k_n \to k$ a.e. and $h_n \to h$ a.e. Furthermore, $|k_n| \leq h_n$ for all n and since $\|f_n\|_p \to \|f\|_p$, we also have $\int h_n \to \int h$. The statement in exercise 21 of the Lebesgue Integration problem list then implies that $\int k_n \to \int k = 0$. In particular, $\|f_n - f\|_p^p \to 0$ and it follows that $\|f_n - f\|_p \to 0$, once the p-th root of the expression is taken.
Note that a similar argument holds for convergence in measure. The first direction of the above proof still holds. For the other direction, appeal to the statement in exercise 10 in this problem list.

23. (a) Note that $L^p(E)$ is a subset of the set of complex valued functions on E, which clearly is a vector space. It therefore suffices to prove that $L^p(E)$ is a subspace. To that end, suppose *f*, g $\in L^p(E)$ and α, β are complex scalars. Since 0 < p < 1, the expression $|\cdot|^p$ defines a metric on **C**, consequently, we have

$$\int |\alpha f + \beta g|^p \leq |\alpha|^p \int |f|^p + |\beta|^p \int |g|^p < \infty,$$

which shows that L^p is closed under function addition and scalar multiplication.

(b) It is easily seen that the expression $d(f, g) = \int |f - g|^p$ is nonnegative with $d(f, g) = 0$ if and only if *f* = g a.e. It is also easily seen that $d(f, g) = d(g, f)$. Triangle inequality follows from the fact that the restriction 0 < p < 1 generates the metric $|\cdot|^p$ on **C**. Monotonicity and additivity of the Lebesgue integral then implies

$$\int |f - g|^p \leq \int \left(|f - h|^p + |h - g|^p\right) \leq \int |f - h|^p + \int |h - g|^p .$$

Thus the function d: $L^p \times L^p \to [0, \infty)$ defines a metric on L^p. The fact that d is translation invariant follows from elementary properties of the integral. Completeness under the metric d can be established by repeating the proof of the Riesz-Fisher theorem presented in exercise 19, where the norm $\|\cdot\|_p$ in the proof must be replaced by the metric d.

(c) The equation $p^{-1} + q^{-1} = 1$ determines that $q^{-1} = (p-1)\,p^{-1}$, which is a negative number, because 0 < p < 1. Thus the assumption g ≥ 0 and $0 < \int g^q < \infty$ must imply g > 0 a.e. Define $\alpha = p^{-1}$ and determine β by the equation $\alpha^{-1} + \beta^{-1} = 1$. Since 0 < p < 1, α must be greater than 1 and consequently α and β are Hoelder conjugates. In particular, $\beta^{-1} = 1 - p$ and therefore $\beta = (1-p)^{-1}$. We can write

$$\int f^p = \int \left(f^p g^p\right) g^{-p} = \int (fg)^p g^{-p} .$$

Applying Hoelder's inequality, we get

$$\int f^p = \int \left(f^p g^p\right) g^{-p} \le \left(\int \left[(fg)^p\right]^\alpha\right)^{1/\alpha} \left(\int \left[g^{-p}\right]^\beta\right)^{1/\beta} = \left(\int fg\right)^p \left(\int g^q\right)^{1-p}.$$

Taking p-th roots on both sides of the inequality gives

$$\left(\int f^p\right)^{1/p} \le \left(\int fg\right) \cdot \left(\int g^q\right)^{-q}.$$

Finally, multiplying both sides by $\left(\int g^q\right)^q$ finishes the proof of the reverse Hoelder inequality.

(d) Suppose f, g $\in L^p$, with f, g ≥ 0. We will derive the reverse Minkowski inequality with help of the reverse Hoelder inequality obtained in part (c). Proceeding through the steps of the Minkowski inequality proof in exercise 11, we obtain

$$\|f+g\|_p^p = \int (f+g)^p = \int f(f+g)^{p-1} + \int g(f+g)^{p-1}.$$

Define q = p/(p−1). Then

$$\int \left[(f+g)^{p-1}\right]^q = \int (f+g)^p < \infty$$

If $\int (f+g)^p = 0$, f + g = 0 a.e. and therefore, since f and g are nonnegative, we must have f = g = 0, in which case the proof of the reverse Minkowski inequality is trivial. Hence, assume without loss of generality that

$$0 < \int (f+g)^p < \infty.$$

Then the hypothesis of the reverse Hoelder inequality of part (c) holds and we have

$$\|f+g\|_p^p = \int (f+g)^p \ge \|f\|_p \left(\int (f+g)^p\right)^{1/q} + \|g\|_p \left(\int (f+g)^p\right)^{1/q},$$

where 1/q = (p−1)/p. In particular,

$$\|f+g\|_p^p = \int (f+g)^p \ge \left(\|f\|_p + \|g\|_p\right) \cdot \|f+g\|_p^{p-1},$$

which simplifies to the desired result, once both sides of the inequality are divided by $\|f+g\|_p^{p-1}$.

(e) Let f, g $L^p(E)$, where 0 < p < 1. We have established in part (b) that $d(f,\ g) = \int |f+g|^p$ is a metric. Consequently,

$$\|f+g\|_p^p = d(f,\ g) \le d(f,\ 0) + d(g,\ 0) = \int \left(|f|^p + |g|^p\right) \le \int 2\max\{|f|^p,\ |g|^p\}.$$

Taking the p-th roots of the leftmost and rightmost expressions, we obtain

$$\|f+g\|_p \le 2^{1/p}\left(\int \max\{|f|^p,\ |g|^p\}\right)^{1/p}. \qquad (1)$$

Define A = {x ∈ E: $|f(x)|^p \ge |g(x)|^p$} and B = {x ∈ E: $|f(x)|^p < |g(x)|^p$}. Then

$$\left(\int \max\{|f|^p,\ |g|^p\}\right)^{1/p} = \left(\int_A |f|^p\right)^{1/p} + \left(\int_B |g|^p\right)^{1/p} \le \|f\|_p + \|g\|_p. \qquad (2)$$

Putting (1) and (2) together shows the desired inequality.

24. First, let us recall the definitions: A function $f: E \to \mathbf{C}$ is said to be **essentially bounded**, if there exists a real number B so that $|f| \le B$ a.e. That is, m{x ∈ E: $|f(x)| > B$} = 0. If U is the collection of all essential bounds of f over E, that is, if U = {M ∈ **R**: m{x ∈ E: $|f(x)| > M$} = 0}, then the **essential supremum** is defined by ess.sup$_{x\in E}$ $|f(x)|$ = inf U. Notice that if m(E) = 0, U = **R** and inf U = -∞. This is not an interesting case from the perspective of Lebesgue measure and integration theory, since the behavior of functions on sets of measure zero has no effect on the integral and can be ignored. We are therefore free to assume m(E) > 0. Throughout this problem, A := inf U.

(a) By hypothesis, f is essentially bounded, which means that U ≠ ∅ and therefore A < ∞. If K < 0, {x ∈ E: $|f(x)| > K$} = E and m{x ∈ E: $|f(x)| > K$} = m(E) > 0. Hence U contains no negative numbers. In particular, A ≥ 0. Finally, notice that A is always a member of U; the set {x ∈ E: $|f(x)| > A$} can be expressed as $\cup_{n\ge1}$ {x ∈ E: $|f(x)| \ge A + 1/n$}, where m{x ∈ E: $|f(x)| \ge A + 1/n$} = 0 (because, by definition of infimum, the interval (A, A +1/(2n)] harbors an element of U). We may therefore conclude that $|f| \le$ A a.e.

(b) Suppose f = 0 a.e. Then m{x ∈ E: $|f(x)| > 0$} = 0 and 0 ∈ U. Since, by part (a), U contains no negative elements, we must have 0 = A.
On the other hand, if A = 0, part (a) implies A ∈ U and we have 0 = m{x ∈ E: $|f(x)| > A$} = m{x ∈ E: $|f(x)| > 0$}, which is the same as saying f = 0 a.e.

(c) If A′ < A, by the definition of infimum, A′ ∈ U and therefore m{x ∈ E: $|f(x)| >$ A′} ≠ 0.

Part (a) of the exercise shows that $|f| \le \|f\|_\infty$ a.e., where $\|f\|_\infty$ = A, while part (c) verifies that $\|f\|_\infty$ = A is the smallest number with this property.

25. Any constant function on E satisfies $\{|f| = \|f\|_\infty\}$ = E. Therefore, m$\{|f| = \|f\|_\infty\}$ > 0 is possible. However, examples where $\{|f| = \|f\|_\infty\}$ = ∅ are abound. Consider, for instance, $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \tan^{-1}(x)$. Then ess.sup $|f|$ = sup $|f|$ = $\pi/2$ as is easily verified. Since this function never attains its least-upper-bound on **R** , it follows that $\{|f| = \|f\|_\infty\}$ = ∅.

26. Let U = {M ∈ **R**: m{x ∈ E: $|f(x)| > M$} = 0} and set A = inf U, B = sup$_E$ $|f|$. Then the hypothesis that f is everywhere bounded by B may be phrased as {x ∈ E: $|f(x)| > B$} = ∅. Hence B ∈ U and therefore A = inf U ≤ B.

The following example shows that ess.sup$_E$ $|f| <$ sup$_E$ $|f|$ can happen:
Define $f: \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & x \text{ irrational} \\ 2 & x \text{ rational} \end{cases}$$

Then m{x ∈ **R**: $|f(x)| > 1$} = m{**Q**} = 0, which shows that ess.sup$_E$ $|f| \le 1$. However, it is clear that sup$_E$ $|f|$ = 2.

27. Define U = {M ∈ **R**: m{x ∈ E: $|f(x)| > M$} = 0}, V = $\left\{ \sup_{x \in E-N} |f(x)|: \ m(N) = 0 \right\}$, A = inf U, and B = inf V. If M ∈ V, there is some 0-measure set N, such that M = $\sup_{x \in E-N} |f(x)|$ and therefore {x ∈ E: $|f(x)| > M$} ⊂ N, which means that M ∈ U. We conclude that V ⊂ U. Consequently, A ≤ B.
To show that A = B, let $\varepsilon > 0$ and set N(ε) = {x ∈ E: $|f(x)| >$ A + ε}. Then m(N(ε)) = 0, because A + ε ∈ U. The number $\sup_{x \in E-N(\varepsilon)} |f(x)|$ must belong in the interval [A, A + ε].
In particular, B ∈ [A, A + ε] and therefore, as ε is arbitrary, A = B.
Finally, to show that ess.sup$_E$ $|f| =$ sup$_{E-N}$ $|f|$ for some set N of measure 0, recall that A = ess.sup$_E$ $|f|$ and that A ∈ U. Let K = {x ∈ E: $|f(x)| >$ A}. Then m(K) = 0 and sup$_{E-K}$ $|f| \le$ A. But sup$_{E-K}$ $|f| \in$ V and therefore B ≤ sup$_{E-K}$ $|f|$. By our earlier observation, A = B and the proof is complete.

28. The fact that f is a continuous function tells us that sets of the form {x ∈ [0, 1]: $|f(x)| >$ A} are open. Hence, if $|f(x)| \le$ A a.e., then {x ∈ [0, 1]: $|f(x)| >$ A} is an open set of measure 0 and can only be the empty set ∅. Thus, in fact, $|f(x)| \le$ A a.e. means $|f(x)| \le$ A for all x. Moreover, if N is any subset of [0, 1] of measure 0,

$$\sup_{0 \le x \le 1} |f(x)| = .\sup_{[0,\ 1]-N} |f(x)|$$

because {x ∈ [0, 1]: $|f(x)| > .\sup_{[0,\ 1]-N} |f(x)|$ } ⊂ N is an open subset of a set of measure 0 and must therefore be the empty set. By the previous exercise, $ess.\sup_{0 \le x \le 1} |f(x)| = .\sup_{[0,\ 1]-N} |f(x)|$ for an appropriately chosen set of measure 0 N. This completes the proof that $\sup_{0 \le x \le 1} |f(x)| = ess.\sup_{0 \le x \le 1} |f(x)|$ in the case where $f \in$ C[0, 1].

29. As has been shown in an earlier exercise, $|f| \le \|f\|_\infty$ a.e. and $|g| \le \|g\|_\infty$ a.e. In particular, the sets F = {x ∈ E: $|f(x)| > \|f\|_\infty$ } and G = {x ∈ E: $|g(x)| > \|g\|_\infty$ } are of measure 0. Since the set H = {x ∈ E: $|f(x)| + |g(x)| > \|f\|_\infty + \|g\|_\infty$ } is a subset of F ∪ G,

which is itself a set of measure 0, it follows that $\|f\|_\infty + \|g\|_\infty$ is an essential upper bound of $|f| + |g|$ and, therefore, of $|f + g|$. Thus, $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$, because $\|f+g\|_\infty$ is the least essential upper bound.

30. Since f, $g \in L^\infty$, $|f| \le \|f\|_\infty$ a.e. and $|g| \le \|g\|_\infty$ a.e. In particular, the sets $F = \{x \in E: |f(x)| > \|f\|_\infty\}$ and $G = \{x \in E: |g(x)| > \|g\|_\infty\}$ are of measure 0 and therefore $|fg| \le \|f\|_\infty \|g\|_\infty$ in the compliment of $F \cup G$. Now $F \cup G$ is a set of measure 0, which means that $\|f\|_\infty \|g\|_\infty$ is an essential upper bound of $|fg|$. Hence $\|f \cdot g\|_\infty \le \|f\|_\infty \|g\|_\infty$, because $\|f \cdot g\|_\infty$ is the least essential upper bound of $|fg|$.
L^∞ is not only a normed algebra, but is also a normed lattice, because L^∞ contains the functions $\min\{|f|, |g|\}$ and $\max\{|f|, |g|\}$ whenever f, $g \in L^\infty$. Notice also that $\|f\|_\infty \le \|g\|_\infty$ whenever $|f| \le |g|$ a.e.

31. Let $f \in L^\infty(E)$, where $m(E) < \infty$. Then $\|f\|_\infty = ess.\sup_{x\in E} |f(x)| = \sup_{x \in E-N} |f(x)|$, where N is some subset of E of measure 0 (see exercise 27). Therefore, we have

$$\begin{aligned} \|f\|_p^p &= \int_E |f|^p = \int_{E-N} |f|^p \\ &\le \int_{E-N} \sup_{x\in E-N} |f|^p = \int_{E-N} \|f\|_\infty^p = m(E-N)\|f\|_\infty^p \\ &= m(E)\|f\|_\infty^p, \end{aligned}$$

which, after taking the p-th root, gives the inequality $\|f\|_p \le m(E)^{1/p}\|f\|_\infty$.

We have thus shown that $f \in L^p(\mathbf{R}^d)$, from whence the set inclusion $L^\infty(E) \subset L^p(\mathbf{R}^d)$ must follow. Combining this result with the one obtained in exercise 12, we conclude that if $f \in L^\infty[0, 1]$, then $\|f\|_1 \le \|f\|_p \le \|f\|_\infty$ for any $1 \le p < \infty$.

32. Let $E \subset \mathbf{R}^d$ be a measurable subset of finite measure and let $f \in L^\infty(E)$. By the result we verified in the previous exercise, we know that $\|f\|_p \le m(E)^{1/p}\|f\|_\infty$. To get a lower estimate for $\|f\|_p$, define the set $H(\varepsilon) = \{|f| > \|f\|_\infty - \varepsilon\}$ and note that, according to exercise 24, $m(H(\varepsilon)) > 0$. By the monotonicity of the integral

$$m(H(\varepsilon))^{1/p} \cdot \left(\|f\|_\infty - \varepsilon\right) \le \left(\int_{H(\varepsilon)} |f|^p\right)^{1/p} \le \|f\|_p.$$

Hence,

$$m(H(\varepsilon))^{1/p} \cdot \left(\|f\|_\infty - \varepsilon\right) \le \|f\|_p \le m(E)^{1/p}\|f\|_\infty. \qquad (1)$$

Observe that $\lim_{p\to\infty} m(H(\varepsilon))^{1/p} = \lim_{p\to\infty} m(E)^{1/p} = 1$. Therefore, taking lim sup of (1), we obtain

$$\|f\|_\infty - \varepsilon \le \limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty,$$

while taking lim inf of (1) yields

$$\|f\|_\infty - \varepsilon \le \liminf_{p\to\infty} \|f\|_p \le \|f\|_\infty .$$

The fact that $\varepsilon > 0$ is arbitrary implies $\limsup_{p\to\infty} \|f\|_p = \liminf_{p\to\infty} \|f\|_p = \|f\|_\infty$. Hence $\lim_{p\to\infty} \|f\|_p = \limsup_{p\to\infty} \|f\|_p = \|f\|_\infty$, which is the desired result.

33. Perhaps the simplest approach is to utilize abstract integration theory by defining a suitable measure function that would make **R** into a measure space of finite measure. With this measure function in place, the problem can be reduced to the one solved in exercise 32 above. Something along the following guidelines was suggested by Prof. Zakeri:
Let M(**R**) be the collection of all Lebesgue measurable subsets of **R**. Define μ: M(**R**) → [0, ∞] by μ(E) = m($\tan^{-1}$(E)). Then μ is a measure function and μ(**R**) = π < ∞. We then have

$$\lim_{n\to\infty}\left(\int_R \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} = \lim_{n\to\infty}\left(\int_R |f|^n\, d\mu\right)^{1/n} = \lim_{n\to\infty} \|f\|_{L^n(\mu)} .$$

Where $L^n(\mu)$ is the space of all L^n-integrable functions on **R** with respect to the measure function μ. Repeating the argument in exercise 31 then yields

$$\lim_{n\to\infty} \|f\|_{L^n(\mu)} = \|f\|_\infty .$$

The problem can also be solved without relying on abstract integration. Observe that since f is essentially bounded, $|f| \le \|f\|_\infty$ a.e. Therefore,

$$\left(\int_R \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} \le \left(\int_R \frac{\|f\|_\infty^n}{1+x^2}\,dx\right)^{1/n} = \pi^{1/n}\|f\|_\infty \qquad (1).$$

Now, to obtain a lower estimate, fix $\varepsilon > 0$ and define for each n the set $H_n(\varepsilon)$ = $\left\{ x \in [-\ln(n),\ \ln(n)] : \frac{|f(x)|}{\sqrt[n]{1+x^2}} > \|f\|_\infty - \varepsilon \right\}$. Then the monotonicity of the integral implies

$$\left(\int_R \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} \ge \left(\int_{H_n(\varepsilon)} (\|f\|_\infty - \varepsilon)^n\,dx\right)^{1/n} = m(H_n(\varepsilon))^{1/n}(\|f\|_\varepsilon - \varepsilon).$$

Thus,

$$m(H_n(\varepsilon))^{1/n}(\|f\|_\varepsilon - \varepsilon) \le \left(\int_R \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} \le \pi^{1/n}\|f\|_\infty \qquad (2).$$

If we can show that $\lim_{n\to\infty} m(H_n(\varepsilon))^{1/n} = 1$, it will follow that

$$\|f\|_\infty - \varepsilon \le \limsup_{n\to\infty}\left(\int_R \frac{|f(x)|^n}{1+x^2}dx\right)^{1/n} \le \|f\|_\infty$$

and that

$$\|f\|_\infty - \varepsilon \le \liminf_{n\to\infty}\left(\int_R \frac{|f(x)|^n}{1+x^2}dx\right)^{1/n} \le \|f\|_\infty .$$

And, as ε is arbitrary, we will then have

$$\liminf_{n\to\infty}\left(\int_R \frac{|f(x)|^n}{1+x^2}dx\right)^{1/n} = \limsup_{n\to\infty}\left(\int_R \frac{|f(x)|^n}{1+x^2}dx\right)^{1/n} = \lim_{n\to\infty}\left(\int_R \frac{|f(x)|^n}{1+x^2}dx\right)^{1/n} = \|f\|_\infty .$$

We now demonstrate that $\lim_{n\to\infty} m(H_n(\varepsilon))^{1/n} = 1$ for all $\varepsilon < \|f\|_\infty$ (if $\|f\|_\infty = 0$, there is nothing to prove). We will do this by appealing to the squeeze theorem. To that end, notice that

$$m(H_n(\varepsilon))^{1/n} \le m[-\ln(n),\ \ln(n)]^{1/n} = (2\ln(n))^{1/n} \qquad (3).$$

To obtain the lower bound expression, define for each integer n the set $K_n(\varepsilon)$ = $\left\{x \in [-\ln(n),\ \ln(n)] : \frac{|f(x)|}{\sqrt[n]{1+[\ln(n)]^2}} > \|f\|_\infty - \varepsilon\right\}$ and observe that since $\lim_{n\to\infty}\sqrt[n]{1+[\ln(n)]^2} = 1$ the sets $K_n(\varepsilon)$ are increasing to the set $K(\varepsilon)$ = $\{x \in [-\ln(n),\ \ln(n)] : |f(x)| > \|f\|_\infty - \varepsilon\}$, which has measure m(K (ε)) > 0 because $\|f\|_\infty - \varepsilon$ is smaller than the least essential upper bound. Furthermore, the inequality

$$\frac{|f(x)|}{\sqrt[n]{1+x^2}} \ge \frac{|f(x)|}{\sqrt[n]{1+[\ln(n)]^2}}\ ;\ x \in H_n(\varepsilon)$$

easily implies $K_n(\varepsilon) \subset H_n(\varepsilon)$. Fix N to be so large that $\sqrt[n]{1+[\ln(n)]^2}\cdot\left(\|f\|_\infty - \varepsilon\right) < \|f\|_\infty$ for all n ≥ N. Then $K_N(\varepsilon) \subset K_n(\varepsilon)$ for all n ≥ N and m($K_N(\varepsilon)$) > 0. In particular,

$$m(K_N(\varepsilon))^{1/n} \le m(H_n(\varepsilon))^{1/n}\ ;\ n \ge N \qquad (4).$$

Combining (3) and (4) we get

$$m(K_N(\varepsilon))^{1/n} \le m(H_n(\varepsilon))^{1/n} \le (2\ln(n))^{1/n} \qquad (5).$$

Taking limit as n → ∞ of (5) shows that $\lim_{n\to\infty} m(H_n(\varepsilon))^{1/n} = 1$ and the proof is complete.